AD680542

ANALYSIS OF THE TURBULENT REGIME OF THE PROGRESS CURVE WHEN NEW LEARNING ADDITIONS HAVE VARIABLE SLOPES

Paul Meyer James

Naval Air Systems Command
Washington, D C

ABSTRACT

Learning curves have been used extensively for predictive purposes in the airframe and other industries. In many instances this has led to erroneous results because analysts failed to extend learning curve theory and develop adequate analytical techniques in the turbulent regime of the cost history characterizing these industries. It is this area where a series of design changes induces a series of perturbations whose turbulence intensity is a function of the frequency of occurrence and magnitudes of the design changes under consideration.

To AD-676 4[illegible]1

In Ref. 1, a series of formulations amenable to machine programming was developed for the accurate determination of perturbed unit costs. This development was based on additions of new learning having a constant slope.

AD-676 4[illegible]1

In this discussion, the development of ~~Ref. 1~~ will be generalized by developing formulas for the addition of new effort having variable slopes. Consideration will also be given to the expressions involving elementary unit cost expressions so that cumulative average and cumulative total values can readily be obtained from existing experience curve tables. Conversely, the problem of determining the magnitudes of design changes and the slopes of new effort from graphical data will also be considered.

INTRODUCTION

The purpose of this investigation is to generalize and extend the expressions developed in Ref. 1. In the interest of generalization, formulas for the addition of new effort having variable slopes will be developed. Conversely, a method will be presented for the confirmation of the slope of the additional new learning and the magnitudes of the design changes imposed. In addition, the analytical expression for the i^{th} perturbation will be expressed in terms of the i design change parameters and $i+1$ elementary functions. The latter consist of the initial unit cost expression and the i unit cost expressions resulting from the additions of new effort associated with the i design changes which follow. With this expression for the i^{th} perturbation in terms of $(i+1)$ unit cost expressions, it will be possible to develop cumulative average and cumulative total cost evaluations based on experience curve tables for use at any point in the turbulent regime.

DERIVATION OF APPLICABLE EQUATIONS

The choice of symbolism in Ref. 1 is inconsistent with that in common usage. The first requirement, therefore, is to re-define quantities so that this inconsistency will be less pronounced. As a start, the initial unit cost expression will be changed to

(1) $$f_0(x) = a_0X^{-b_0}$$

where

a_0 = cost of first production item

b_0 = numerical value of exponent

x = production quantity.

The subscript zero in the foregoing characterizes cost conditions in the unperturbed regime.

The turbulent cost regime, on the other hand, is characterized by a series of cost discontinuities induced by a series of major design changes. The degree of turbulence is a function of the frequency of occurrence of these design changes and their magnitudes.

The formulas associated with the turbulent cost regime are characterized by the subscript i where $i = 1, 2, \ldots, n$, when evaluated, is indicative of a particular perturbation caused by a particular design change. The i^{th} perturbation will become effective after the completion of item X_i and continue in effect until item X_{i+1} is completed. Associated with the i^{th} major design change are two weighting factors λ_i and τ_i along with the new parameter, b_i, where the latter is indicative of variable new learning capabilities. λ_i represents the present reduction in the previous manufacturing effort, $f_{i-1}(X - X_{i-1})$, while τ_i represents the percent of new effort to be added. The unit cost expression, $f_i(X - X_i)$, after the i^{th} discontinuity in the range, $X_i < X \leq X_{i+1}$, is given by the following recursion formula:

(2) $$f_i(X - X_i) = (1 - \lambda_i) f_{i-1}(X - X_{i-1}) + \tau_i a_0 (X - X_i)^{-b_i}$$

where $(i = 1, 2, \ldots, n)$ and $X_0 = 0$.

It is now possible to state the expression for the i^{th} perturbation in terms of the i design change parameters and i + 1 elementary functions. As indicated previously the latter consist of the initial unit cost expression and the i unit cost expressions resulting from the additions of variable new effort associated with the i design changes which follow. By using Eqs. (1) and (2) and substituting $f_{i-1}(X - X_{i-1})$ into $f_i(X - X_i)$, it is apparent that for i = 1, 2, 3, etc., we have:

(3) $$f_1(X - X_1) = (1 - \lambda_1) f_0(X) + \tau_1 a_0 (X - X_1)^{-b_1}$$

$$= (1 - \lambda_1) a_0 X^{-b_0} + \tau_1 a_0 (X - X_1)^{-b_1}$$

(4) $$f_2(X - X_2) = (1 - \lambda_2) f_1(X - X_1) + \tau_2 a_0 (X - X_2)^{-b_2}$$

$$= (1 - \lambda_2)\left[(1 - \lambda_1) a_0 X^{-b_0} + \tau_1 a_0 (X - X_1)^{-b_1}\right] + \tau_2 a_0 (X - X_2)^{-b_2}$$

$$= (1 - \lambda_2)(1 - \lambda_1) a_0 X^{-b_0} + (1 - \lambda_2) \tau_1 a_0 (X - X_1)^{-b_1} + \tau_2 a_0 (X - X_2)^{-b_2}$$

$$f_3(X - X_3) = (1 - \lambda_3) f_2(X - X_2) + \tau_3 a_0 (X - X_3)^{-b_3} \tag{5}$$

$$= (1 - \lambda_3) \left[(1 - \lambda_2)(1 - \lambda_1) a_0 X^{-b_0} + (1 - \lambda_2) \tau_1 a_0 (X - X_1)^{-b_1} + \tau_2 a_0 (X - X_2)^{-b_2} \right] + \tau_3 a_0 (X - X_3)^{-b_3}$$

$$= (1 - \lambda_3)(1 - \lambda_2)(1 - \lambda_1) a_0 X^{-b_0} + (1 - \lambda_3)(1 - \lambda_2) \tau_1 a_0 (X - X_1)^{-b_1} + (1 - \lambda_3) \tau_2 a_0 (X - X_2)^{-b_2} + \tau_3 a_0 (X - X_3)^{-b_3}$$

Equation (5) can be expressed more compactly as

$$f_3(X - X_3) = C_3 (X - X_0)^{-b_0} + C_2 (X - X_1)^{-b_1} + C_1 (X - X_2)^{-b_2} + \tau_3 a_0 (X - X_3)^{-b_3} . \tag{6}$$

Equation (6) is a special case of the following expression:

$$f_i(X - X_i) = \sum_{j=0}^{j=i-1} C_{i-j} (X - X_j)^{-b_j} + \tau_i a_0 (X - X_i)^{-b_i} , \tag{7}$$

where

$$C_{i-j} = a_0 \tau_j \prod_{k=j+1}^{k=i} (1 - \lambda_k), \tag{8}$$

and

$$\tau_0 = 1. \tag{9}$$

By substitution of Eq. (8) in Eq. (7), it is seen that

$$f_i(X - X_i) = a_0 \sum_{j=0}^{j=i-1} \tau_j \prod_{k=j+1}^{k=i} (1 - \lambda_k)(X - X_j)^{-b_j} + \tau_i a_0 (X - X_i)^{-b_i} \tag{10}$$

and

$$f_i(X - X_i) = a_0 \sum_{j=0}^{j=i-1} \tau_j (X - X_j)^{-b_j} \prod_{k=j+1}^{k=i} (1 - \lambda_k) + \tau_i a_0 (X - X_i)^{-b_i} . \tag{11}$$

In connection with the foregoing development, attention is invited to the fact that either Eq. (7) in conjunction with Eq. (8) or their composite equivalent, Eq. (11), is an expression for

the i^{th} perturbation in terms of the initial unit cost expression, Eq. (1), and the i unit cost expressions associated with the i design changes which follow. The latter, it will be noted, become effective with production quantities $X_i + 1$ where $i = 1, 2, \ldots, n$ respectively. Equations (5), (4) and (3) are evaluations of Eq. (11) or Eqs. (7) and (8) for $i = 3, 2, 1$ respectively. A verification of Eq. (11) is given in Appendix A.

Since Eqs. (7) and (8) or the composite relationship, Eq. (11), involve $(i+1)$ unit cost expressions, the corresponding cumulative average and cumulative total values can readily be obtained from existing experience curve tables. The latter are particularly useful in arriving at an equitable financial adjustment for the extension of a contract and the procurement of additional production quantities in the turbulent regime.

DETERMINATION OF NEW EFFORT SLOPE AND DESIGN CHANGE PARAMETERS FROM GRAPHICAL DATA

The problem of determining the magnitudes of the slope and design change parameters from graphical data will now be discussed. A requirement for the determination is that these quantities must be known for each of the preceding i perturbations along with the slope of the initial unit cost expression. In cases where this information is not available, it will be necessary to start with the graph of the initial unit cost expression and determine the design change parameters and new learning slope associated with the first perturbation. With this information, the same procedure can be applied to the 2nd, 3rd, and finally to the (i + 1st) perturbation. The previously derived expressions can then be used in conjunction with a technique of successive approximations.

As an illustration of the foregoing, a method of determining the design change parameters will be illustrated by considering for simplicity a single perturbation. It is assumed that the initial cost of an item, a_0, is 100 and that $b_0 = 0.322$ for 80% learning. After the completion of production item $X = 30$, a design change calling for $\lambda_1 = 0.113$ and $\tau_1 = 0.207$ becomes effective. The value of b_1 is assumed to be 0.454 for 73% learning.

The foregoing design change is represented graphically on the linear scale, Figure 1, and the conventional log-log scale, Figure 2. Without being influenced by these design changes, the problem now is to use graphical values, and compute values of λ_1, τ_1 and b_1. With this in mind, the coordinates of three points read from the graph had the following values:

X	$f_1(X - X_1)$
31	50.0
32	44.2
34	39.8

When these three sets of values are substituted in the governing equation

$$f_1(X - X) = (1 - \lambda_1) a_0 X^{-b_0} + a_0 \tau_1 (X - X_1)^{-b_1} ,$$

three equations in three unknowns are obtained. Because of the transcendental nature of this set of equations together with the round-off errors and errors resulting from using graphical values, the results of the computations given in Appendix B left something to be desired.

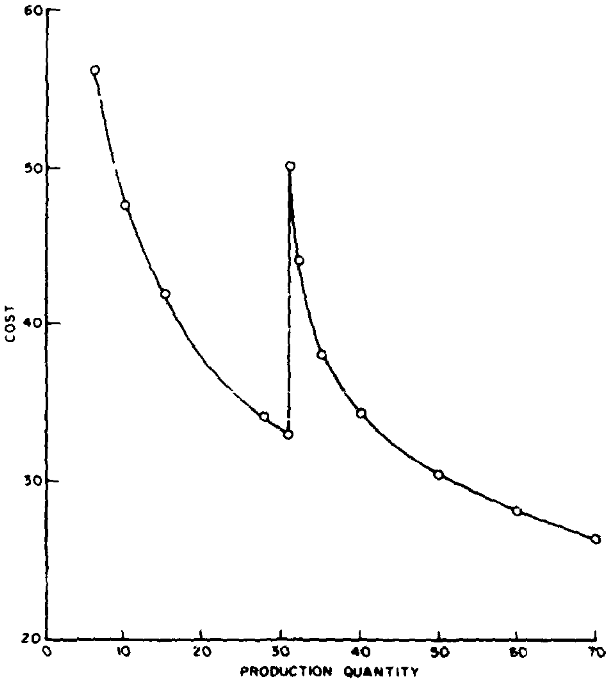

Figure 1. Linear scale graph

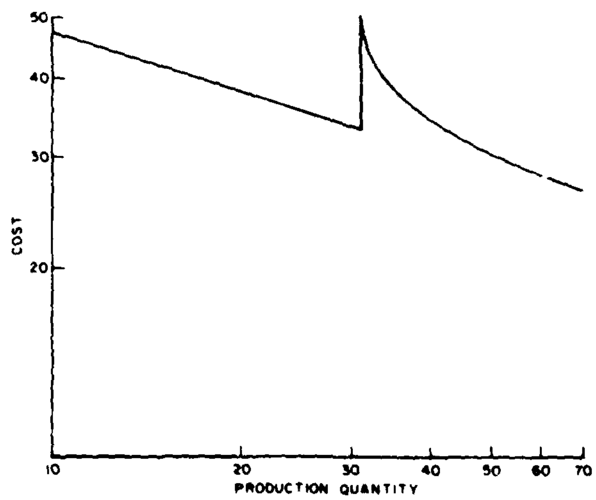

Figure 2. Log-log scale graph

In an effort to obtain acceptable accuracies in the computation of λ_1, τ_1, and b_1, it was necessary to resort to a method of successive approximations. In this connection, the applicable expression $f_1(X - X_1)$ is unitized and rearranged so that the new effort is expressed as a function of the graphical value, the initial unit cost expression, and the parameter λ_1. Points are read from the graph in the neighborhood of the discontinuity and one, a check point, is taken as far from the discontinuity as possible. By inserting trial values of $\lambda_1 = 0.05$, 0.10, 015, etc., corresponding slopes, $S = 2^{-b_1}$, are obtained. On the basis of the slope which produces a prediction whose deviation from the extreme check point is a minimum, values of λ_1 and τ_1 are computed. By using values of λ_1, in the neighborhood of the value just computed, the program is repeated until convergence develops. Details of the method follow:

(12) $$f_1(X - X_1) = (1 - \lambda_1) a_0 X^{-b_0} + a_0 \tau_1 (X - X_1)^{-b_1},$$

(13) $$g_1(X - X_1) = f_1(X - X_1)/a_0 = (1 - \lambda_1) X^{-b_0} + \tau_1 (X - X_1)^{-b_1},$$

and

(14) $$\tau_1 (X - X_1)^{-b_1} = g_1(X - X_1) - (1 - \lambda_1) X^{-b_0},$$

where $\tau_1\ X - X_1^{\ -b_1}$ is the new effort in terms of the graphical value, $g_1(X - X_1)$, the original unitized unit cost expression, X^{-b_0} and the parameter λ_1. By using values X, X'', X''' where $X' = X_1 + 1$, $X'' = X_1 + 2$, $X''' = X_1 + 4$, Eq. (14) produces the following:

(15) $$g_1(X' - X_1) - (1 - \lambda_1)(X')^{-b_0} = \tau_1,$$

(16) $$g_1(X'' - X_1) - (1 - \lambda_1)(X'')^{-b_0} = \tau_1(2)^{-b_1},$$

and

(17) $$g_1(X''' - X_1) - (1 - \lambda_1)(X''')^{-b_0} = \tau_1(4)^{-b_1}.$$

Equations (15), (16), and (17) are expressed in tabular form for $\lambda_1 = 0.05$, 0.10, 0.15, respectively, in the following Table 1:

TABLE 1

(1)	(2)	(3)	(4)	(5)	(6)	(7) (2) - (4)	(8) (2) - (5)	(9) (2) - (6)
X	$g_1(X - X_1)$	X^{-b_0}	$0.95X^{-b_0}$	$0.90X^{-b_0}$	$0.85X^{-b_0}$	$\lambda_1 (X - X_1)^{-b_1}$ $\lambda_1 = 0.05$	$\lambda_1 (X - X_1)^{-b_1}$ $\lambda_1 = 0.10$	$\lambda_1 (X - X_1)^{-b_1}$ $\lambda_1 = 0.15$
31	0.500	0.3311	0.3145	0.2980	0.2814	0.1855	0.2020	0.2186
32	0.442	0.3277	0.3177	0.2949	0.2785	0.1307	0.1471	0.1635
34	0.398	0.3214	0.3053	0.2893	0.2732	0.0927	0.1087	0.1248
40	0.345	0.3050	0.2858	0.2745	0.2592	0.0552	0.0705	0.0858
70	0.265	0.2547	0.2420	0.2292	0.2165	0.0230	0.0358	0.0485

Division of Eq. (16) by (15) produces the slope $S = 2^{-b_1}$. The slope numerical values for λ_1 = 0.05, 0.10, 0.15, respectively, are obtained by dividing the second line by the first in columns (7), (8) and (9) of Table 1, therefore, Table 2.

TABLE 2
(PARTS 1 AND 2)

PART 1					
(10)	(11)	(12)	(13)	(14)	
1	$S = \frac{f_2}{f_1} = 2^{-b_1}$	$0.1855\left(X - X_1\right)^{-b_1}$ $\lambda_1 = 0.05$ $X_1 = 0.30$; $X = 40$	$0.2020\left(X - X_1\right)^{-b_1}$ $\lambda_1 = 0.10$ $X_1 = 0.30$; $X = 40$	$0.2186\left(X - X_1\right)^{-b_1}$ $\lambda_1 = 0.15$ $X_1 = 0.30$; $X = 40$	Comments
0.05	0.7046	0.0567			Columns (12), (13) and (14) are predictions based on slopes, col. (11) for λ_1 = 0.05, 0.10, and 0.15
0.10	0.7272		0.0710		
0.15	0.7479			0.0841	
PART 2					
0.05		0.0552			These values were taken from Cols. (7), (8), and (9) with X = 40
0.10			0.0705		
0.15				0.0858	
		0.0015	0.0005	-0.0017	

In the first section of the foregoing table covering columns (10) to (14) predictions are made for the cost of the new effort at X = 40 for slopes of 0.7046, 0.7282 and 0.7479 based on initial trial values of λ_1 = 0.05, 0.10, 0.15 respectively. The coefficients of the negative exponential in the new effort expressions, columns (12), (13) and (14) are the numbers given on the first line under columns (7), (8) and (9). The values appearing in the second section of this table under columns (12), (13) and (14) are the values for X = 40, line 4, under columns (7), (8) and (9). From the differences given in the last line, it is apparent that the slope lies between 0.7282 and 0.7497. The value 0.74 will be used for the second iteration.

By substituting values for X = 31 and 32 under columns (2) and (3) into Eqs. (15) and (16), we obtain

(18) $$0.500 - 0.3311(x - \lambda_1) = \tau$$

and

(19) $$0.442 - 0.3277(1 - \lambda_1) = (1.74)\,\tau_1$$

with the solution being given by $\tau_1 = 0.2118$ and $\lambda_1 = 0.1294$.

The second iteration was based on λ_1 values of 0.10, 0.13 and 0.16. These were used in columns (4), (5) and (6) respectively and the entire computation repeated. At the end of the second iteration, the following numerical values were obtained: $S = 2^{-b_1} = 0.736$, $\tau_1 = 0.2085$ and $\lambda_1 = 0.121$.

The third iteration was based on λ_1 values in the neighborhood of 0.121 which was the result of the second iteration. The actual λ values substituted in columns (4), (5) and (6) were 0.11, 0.12 and 0.13. This iteration produced the following values: $S = 2^{-b_1} = 0.732$, $\tau_1 = 0.2053$ and λ_1 (average) = 0.112. The latter, it will be observed, compares very favorably with the true values of $S = 2^{-b_1} = 0.73$, $\tau_1 = 0.207$ and $\lambda_1 = 0.113$.

RESULTS

The formulations derived herein are suitable for accurately describing the entire history of a production item in the turbulent regime when additions of new learning are individually different; that is, the formulations account for i different learning rates when i design changes are specified. These equations have also been expressed in terms of (i + 1) elementary unit cost expressions so that cumulative average and cumulative total values can readily be obtained from existing tables for contract adjustments. By using a technique of successive approximations, the expressions can be used in conjunction with graphical data for the determination of not only the design change parameters but the learning rate associated with the addition of new effort when a design change has been made. All formulations are well suited to machine programming efforts and repetitive computational procedures.

Appendix A

VERIFICATION OF EQUATION (11)

$$f_i(X - X_i) = a_0 \sum_{j=0}^{j=i-1} \tau_j (X - X_j)^{-b_j} \prod_{k=j+1}^{k=i} (1 - \lambda_k) + \tau_i a_0 (X - X_i)^{-b_i}.$$

Letting i = 3, it is seen that

$$f_3(X - X_3) = a_0 \left\{ \tau_0 (X - X_0)^{-b_0} \prod_{k=1}^{k=3} (1 - \lambda_k) + \tau_1 (X - X_1)^{-b_1} \prod_{k=2}^{k=3} (1 - \lambda_k) \right.$$

$$\left. + \tau_2 (X - X_2)^{-b_2} \prod_{k=3}^{i=3} (1 - \lambda_k) \right\} + \tau_3 a_0 (X - X_3)^{-b_3}.$$

(1A)
$$f_3(X - X_3) = a_0 \tau_0 (X - X_0)^{-b_0} (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)$$
$$+ a_0 \tau_1 (X - X_1)^{-b_1} (1 - \lambda_2)(1 - \lambda_3)$$
$$+ a_0 \tau_2 (X - X_2)^{-b_2} (1 - \lambda_3)$$
$$+ a_0 \tau_3 (X - X_3)^{-b_3}$$

In view of the fact that $\tau_0 = 1$ by Eq. (9), it is apparent that Eq. (1A) is the same as Eq. (11).

Appendix B

COMPUTATIONS USING GRAPHICAL VALUES

(1B)
$$f_1(X - X_1) = (1 - \lambda_1) a_0 X^{-b_0} + a_0 \tau_1 (X - X_1)^{-b_1}$$

(2B)
$$g_1(X - X_1) = f_1(X - X_1)/a_0 = (1 - \lambda_1) X^{-b_0} + \tau_1 (X - X_1)^{-b_1}$$

X	$f_1(X - X_1)$	$f_1(X - X_1)/a_0$	X^{-b_0}
31	50.0	0.500	0.3310
32	44.2	0.442	0.3277
34	39.8	0.398	0.3214

Substitution of the above tabular values in Eq. (2B) produces.

$$0.5000 = 0.3310(1 - \lambda_1) + \tau_1 (1)^{-b_1}$$
$$0.4420 = 0.3277(1 - \lambda_1) + \tau_1 (2)^{-b_1}$$
$$0.3980 = 0.3214(1 - \lambda_1) + \tau_1 (4)^{-b_1}$$

and

$$0.1690 = -0.3310 \lambda_1 + \tau_1$$
$$0.1143 = -0.3277 \lambda_1 + \tau_1 (2)^{-b_1}$$
$$0.0766 = -0.3214 \lambda_1 + \tau_1 (4)^{-b_1}$$

(3B) $$0.5106 = -\lambda_1 + 3.0211\,\tau_1$$

(4B) $$0.3488 = -\lambda_1 + 3.0516\,(2^{-b_1})\,\tau_1$$

(5B) $$0.2383 = -\lambda_1 + 3.1114\,(4^{-b_1})\,\tau_1$$

(6B) $$(3B)-(4B):\ 0.1618 = 3.0211\,\tau_1 - 3.0516\,(2^{-b_1})\,\tau_1$$

(7B) $$(3B)-(5B):\ 0.2723 = 3.0211\,\tau_1 - 3.1114\,(4^{-b_1})\,\tau_1$$

From Eqs. (6B) and (7B), it is seen that

(8B) $$\tau_1 = \frac{0.1618}{3.0211 - 3.0516\,(2^{-b_1})} = \frac{0.2723}{3.0211 - 3.1114\,(4^{-b_1})}$$

and

$$0.1618\left[3.0211 - 3.1114\,(4^{-b_1})\right] = 0.2723\left[3.0211 - 3.0516\,(2^{-b_1})\right]$$

or

$$0.1105(3.0211) = 0.2723\left[3.0516\,(2^{-b_1})\right] - 0.1618\left[3.1114\,(4^{-b_1})\right]$$

(9B) $$0.3338 = 0.8310\,\omega - 0.5034\,\omega^2$$

where $\omega = (2^{-b_1})$ and $\omega^2 = (4^{-b_1})$.

Solving the quadratic, Eq. (9B), it is seen that

$$\omega = 0.8254 \pm 0.1348 = 0.6906 \text{ or } .9602.$$

Since a 73% learning was used, 0.69/0.73 = 0.945 or 5.5% low.

By using $\omega = (2^{-b_1}) = 0.69$ in Eqs. (3B) and (4B), we obtain the following:

$$0.5106 = -\lambda_1 + 3.0211\,\tau_1$$

and

$$0.3488 = -\lambda_1 + 2.1056\,\tau_1.$$

$\tau_1 = 0.1618/0.9155 = 0.1767$ which is 0.177/0.207 = 0.855 or 14.5% low and $\lambda_1 = 0.1044$ which is 0.104/0.113 = 0.920 or 8% low.

REFERENCE

[1] James, P. M., "Derivation and Application of Unit Cost Expression Perturbed by Design Changes," Nav. Res. Logist. Quart. 15, 459-468 (1968).

* * *